

HEEGNER POINTS AND THE RANK OF ELLIPTIC CURVES OVER LARGE EXTENSIONS OF GLOBAL FIELDS

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ABSTRACT. Let k be a global field, \bar{k} a separable closure of k , and G_k the absolute Galois group $\text{Gal}(\bar{k}/k)$ of \bar{k} over k . For every $\sigma \in G_k$, let \bar{k}^σ be the fixed subfield of \bar{k} under σ . Let E/k be an elliptic curve over k . We show that for each $\sigma \in G_k$, the Mordell-Weil group $E(\bar{k}^\sigma)$ has infinite rank in the following two cases. Firstly when k is a global function field of odd characteristic and E is parametrized by a Drinfeld modular curve, and secondly when k is a totally real number field and E/k is parametrized by a Shimura curve. In both cases our approach uses the non-triviality of a sequence of Heegner points on E defined over ring class fields.

1. INTRODUCTION

This paper is motivated by the following conjecture of M. Larsen (see [14] for context):

Conjecture. Let E/k be an elliptic curve over a finitely generated infinite field k . Then, for every $\sigma \in G_k := \text{Gal}(\bar{k}/k)$, the Mordell-Weil group $E(\bar{k}^\sigma)$ of E over $\bar{k}^\sigma = \{x \in \bar{k} \mid \sigma(x) = x\}$ has infinite rank.

Larsen [14] has shown that the conjecture is true for σ in a suitable open subset of G_k . In [10], [11] and [12], the conjecture is proved for the following cases over number fields:

- (1) If E/k has a k -rational point P such that $2P \neq O$ and $3P \neq O$, or
- (2) if the 2-torsion points of E/k are k -rational, or
- (3) if $k = \mathbb{Q}$ without any hypothesis on rational points,

then, for every automorphism $\sigma \in G_k$, the rank of the Mordell-Weil group $E(\bar{k}^\sigma)$ is infinite.

In fact, under the first assumption above it is shown in [10] that for each $\sigma \in \text{Gal}(\bar{k}/k)$, the rank of E over the maximal Galois extension of k in \bar{k}^σ is infinite. In general, the maximal Galois extension of k in \bar{k}^σ is smaller than \bar{k}^σ . Moreover, under the second assumption above that the 2-torsion points of E/k are k -rational, it was shown in [11] that E has infinite rank over the maximal abelian extension of k in \bar{k}^σ , which is also much smaller.

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In this paper, we prove that Larsen's conjecture is true for certain modular elliptic curves over global fields. When k is a totally real number field, We say E/k is modular if E/k is parametrized by a suitable Shimura curve. Every elliptic curve over $k = \mathbb{Q}$ is parametrized by a modular curve $X_0(N)$, where N is the conductor of E/\mathbb{Q} , by [18], [17] and [4]. $X_0(N)$ is a particularly simple Shimura curve, so E/\mathbb{Q} is also modular in the above sense. When k is a global function field and E/k has split multiplicative reduction at a place ∞ , then the conductor of E/k may be written as $\mathfrak{n} \cdot \infty$, where \mathfrak{n} is an ideal in the Dedekind ring $A := \{x \in k \mid x \text{ is regular away from } \infty\}$, and E/k is parametrized by the Drinfeld modular curve $X_0(\mathfrak{n})$, see [7].

Our approach is the following. Let E/k be a modular elliptic curve, then for a given automorphism $\sigma \in G_k$, we produce an infinite sequence K_m of imaginary quadratic extensions of k (if k is a function field, then K_m/k is *imaginary* if ∞ does not split in K_m) such that

- (1) if $\sigma|_{K_m} = \text{id}_{K_m}$ for all m , then we may show by elementary arguments that the rank of $E(\overline{k}^\sigma)$ is infinite and
- (2) if $\sigma|_{K_m} \neq \text{id}_{K_m}$ for some m , then (E, K_m) satisfies the *Heegner hypothesis* (the definition is given below). This allows us to construct a suitable sequence of Heegner points on E defined over a tower of ring class fields of K_m over which the rank of E is unbounded. Then we use the dihedral structure of these ring class fields to show that the rank of $E(\overline{k}^\sigma)$ is infinite.

Note that for number fields we simplify the proof of the results of [12], using an argument from [2].

If k is a function field and E/k has non-constant j -invariant, then there exists a finite extension L/k such that E/L has split multiplicative at some place of L , and hence our results apply to E/L , but this case is already covered by [14, Theorem 5].

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2. MODULAR ELLIPTIC CURVES OVER GLOBAL FUNCTION FIELDS

Throughout this section, we let k be a global function field with field of constants \mathbb{F}_q , where q is a power of the odd prime p , and we let A be the ring of functions in k regular outside the place ∞ . Let E/k be an elliptic curve over k with split multiplicative reduction at ∞ , then the conductor of E/k is $\mathfrak{n} \cdot \infty$ for an ideal $\mathfrak{n} \subset A$.

Denote by $X_0(\mathfrak{n})$ the Drinfeld modular curve parametrizing pairs of rank-2 Drinfeld A -modules linked by cyclic \mathfrak{n} -isogenies. Then we have a morphism defined over k (see [7]):

$$\pi : X_0(\mathfrak{n}) \longrightarrow E.$$

Let K/k be a quadratic *imaginary* extension, i.e. ∞ does not split in K/k . Suppose that all prime divisors of \mathfrak{n} split in K/k , we say that (E, K) satisfies the **Heegner hypothesis**.

We fix a prime \mathfrak{p} of A not dividing \mathfrak{n} .

Denote by \mathcal{O}_K the integral closure of A in K , then thanks to the Heegner hypothesis, there exists an ideal $\mathcal{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathcal{N} \cong A/\mathfrak{n}$. For every non-negative integer n we denote by $\mathcal{O}_n := A + \mathfrak{p}^n \mathcal{O}_K$ the order in \mathcal{O}_K of conductor \mathfrak{p}^n , and set $\mathcal{N}_n := \mathcal{N} \cap \mathcal{O}_n$. We have $\mathcal{O}_n/\mathcal{N}_n \cong A/\mathfrak{n}$ for all $n \geq 0$. Denote by $\mathbb{C}_\infty = \hat{k}_\infty$ the completion of an algebraic closure of the completion of k at ∞ , a field both algebraically closed and complete, which plays the role of the complex numbers in characteristic p . Then \mathcal{O}_K and \mathcal{N}_n^{-1} are rank-2 lattices in \mathbb{C}_∞ , hence define a pair of Drinfeld modules $(\Phi^{\mathcal{O}_K}, \Phi^{\mathcal{N}_n^{-1}})$ linked by a cyclic \mathfrak{n} -isogeny. The pair thus defines a point x_n on $X_0(\mathfrak{n})$, which is defined over the ring class field $K[\mathfrak{p}^n]$ of the order \mathcal{O}_n . Its image $y_n = \pi(x_n) \in E(K[\mathfrak{p}^n])$ is called a *Heegner point* on E .

Recall that $\text{Gal}(K[\mathfrak{p}^n]/K) \cong \text{Pic}(\mathcal{O}_n)$. We now show that the rank of $E(K[\mathfrak{p}^n])$ is unbounded as $n \rightarrow \infty$. This follows from [3, Theorem 4.4], but notice that in that paper and in [1] the Heegner points y_n are constructed differently, involving a trace, and consequently the proof is more difficult. For our purposes the following result for our Heegner points is sufficient, and much easier.

Proposition 2.1. *Let $K[\mathfrak{p}^\infty] = \bigcup_{n \geq 0} K[\mathfrak{p}^n]$. Let $I \subset \mathbb{N}$ be an infinite set. Then the subgroup of $E(K[\mathfrak{p}^\infty])$ generated by $\{y_n \mid n \in I\}$ has finite torsion and infinite rank.*

Proof. From [1, Lemma 2.2] follows that $E(K[\mathfrak{p}^\infty])$ has finite torsion, so by [12, Lemma 2.5], it remains to show that the subgroup $H \subset E(K[\mathfrak{p}^\infty])$ generated by the y_n 's is not finitely generated. For this we use the same argument as in [2].

Suppose that H is finitely generated. Then $H \subset E(L)$ for some finite separable extension L/k , which we may extend to include K . Denote by $G_L = \text{Gal}(\bar{L}/L)$ the absolute Galois group of L . Then G_L acts on the fibers $\pi^{-1}(y_n)$, and the G_L -orbit of x_n is bounded: $\#G_L \cdot x_n \leq \deg(\pi)$.

On the other hand,

$$\begin{aligned} \#G_L \cdot x_n &\geq \#\text{Pic}(\mathcal{O}_n)/[L : K] \\ &\geq \frac{\#\text{Pic}(\mathcal{O}_K)}{[L : K][\mathcal{O}_K^\times : \mathcal{O}_n^\times]} |\mathfrak{p}^n| (1 - |\mathfrak{p}|^{-1}) \\ &\geq \frac{\#\text{Pic}(\mathcal{O}_K)}{[L : K](q + 1)} |\mathfrak{p}|^n (1 - |\mathfrak{p}|^{-1}) \end{aligned}$$

by equation (2.5) of [3]. This is unbounded as n gets large, which is a contradiction. \square

We point out that everything up to now holds almost verbatim if we replace the function field k with a number field, provided that E/k is parametrized by the (classical) modular curve $X_0(N)$. In particular, this provides a simplified proof of the conclusion of [12, Proposition 2.7].

We will need the following result later.

Proposition 2.2. *For positive integers n and m such that $n > m$, the Galois group $\text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^m])$ of the ring class fields $K[\mathfrak{p}^n]$ over $K[\mathfrak{p}^m]$ is a finite direct sum of cyclic groups of order p , where p is the characteristic of k , and the Galois group $\text{Gal}(K[\mathfrak{p}^n]/k)$ over k is generalized dihedral as in the number field case.*

Proof. See [5, Sec. 2.3 and 2.5, Proposition 2.5.7]. \square

Finally, the following proposition proves the infinite rank of E over K_{ab}^σ , which will allow us to complete one of our main results, Theorem 3.2, which is introduced in the next section.

Proposition 2.3. *Let $\sigma \in G_k$ and K a quadratic imaginary extension of k such that $\sigma|_K \neq id_K$ and ∞ does not split in K . Let E/k be an elliptic curve over k with split multiplicative reduction at ∞ . Suppose all primes dividing \mathfrak{n} split in K , where $\mathfrak{n} \cdot \infty$ is the conductor of E/k . Let $\mathfrak{p} \subset A$ be a prime not dividing \mathfrak{n} . Then, the rank of the Mordell-Weil group $E((K[\mathfrak{p}^n])^\sigma)$ over the fixed subfield of $K[\mathfrak{p}^n]$ under σ is unbounded, as n goes to ∞ . In particular, the rank of $E(K_{ab}^\sigma)$ is infinite, where K_{ab} is the maximal abelian extension of K .*

Proof. For the given $\sigma \in G_k$, since $\sigma|_K \neq id_K$, the restriction of σ to each ring class field $K[\mathfrak{p}^n]$ of conductor n can be lifted as an involution of $K[\mathfrak{p}^n]$. Let $\sigma_n = \sigma|_{K[\mathfrak{p}^n]}$ be the restriction of σ to $K[\mathfrak{p}^n]$. Then, since the Galois group of each ring class field $K[\mathfrak{p}^n]$ over k has a generalized dihedral group structure by Proposition 2.2, σ_n acts on $K[\mathfrak{p}^n]$ by an involution such that

$$(*) \quad \text{for any } \tau \in \text{Gal}(K[\mathfrak{p}^n]/k), \sigma_n \tau \sigma_n = \tau^{-1}.$$

Now, we prove that the rank of $E(K[\mathfrak{p}^n]^\sigma)$ is unbounded as n goes to infinity. Suppose not. Then since the restriction σ_n of σ acts by an involution on each ring class field $K[\mathfrak{p}^n]$, and by Proposition 2.1 the rank of $E(K[\mathfrak{p}^n])$ is unbounded as n goes to infinity, there exists a fixed integer n_0 such that σ acts by -1 on any nontrivial quotient $M_n := E(K[\mathfrak{p}^n])/E(K[\mathfrak{p}^{n_0}])$, for all $n > n_0$. Since $K[\mathfrak{p}^{n_0}]$ is Galois over k , $\text{Gal}(K[\mathfrak{p}^n]/k)$ acts on $E(K[\mathfrak{p}^{n_0}])$ by restriction to $K[\mathfrak{p}^{n_0}]$, so it does act on M_n .

Let

$$\rho : \text{Gal}(K[\mathfrak{p}^n]/k) \rightarrow \text{Aut}(M_n)$$

be the representation of $\text{Gal}(K[\mathfrak{p}^n]/k)$. Then, by the hypothesis, σ_n acts by -1 on M_n . Hence, $\rho(\sigma_n) = -id$ on M_n . Therefore,

$$(**) \quad \rho(\tau^2) = \rho(\tau)\rho(\tau) = (-id)\rho(\tau)(-id)\rho(\tau) = \rho(\sigma_n \tau \sigma_n \tau) \stackrel{by(*)}{=} \rho(1) = id.$$

In particular, for $\tau \in \text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])$, the restriction of ρ to the subgroup $\langle \tau^2 \rangle$ of $\text{Gal}(K[\mathfrak{p}^n]/k)$ generated by the element τ^2 for each $\tau \in \text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])$ is a trivial representation of M_n . By Proposition 2.2, $\text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])$ is a p -group with the odd prime p which is the characteristic of k . So the order of every nontrivial element in $\text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])$ is the odd prime p . So $\langle \tau^2 \rangle = \langle \tau \rangle$. Therefore, $(**)$ implies that $\rho|_{\langle \tau \rangle} = id$. So we have that

$$M_n = M_n^{\langle \tau \rangle}, \text{ for all } \tau \in \text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}]).$$

Since this is true for all $\tau \in \text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])$, this implies that

$$M_n = M_n^{\text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])}.$$

So

$$E(K[\mathfrak{p}^n]) = E(K[\mathfrak{p}^n]^{\text{Gal}(K[\mathfrak{p}^n]/K[\mathfrak{p}^{n_0}])}) = E(K[\mathfrak{p}^{n_0}]), \text{ for all } n > n_0,$$

which is a contradiction to Proposition 2.1.

Therefore, the rank of $E(K[\mathfrak{p}^n]^\sigma)$ is unbounded, as $n \rightarrow \infty$. Since all ring class fields $K[\mathfrak{p}^n]$ are abelian over K , this implies that the rank of $E((K_{ab})^\sigma)$ is infinite. \square

3. INFINITE RANK IN ODD CHARACTERISTIC p

In this section, we prove our main theorem for the global function field case. As a generalization of [16, Lemma] (which is made in the number field setting) for global function fields, we will need the following lemma to prove the linear independence of algebraic points defined over extensions of bounded degree.

Lemma 3.1. *Let k be a global function field with characteristic p and E/k an elliptic curve over k . Then for any integer $d > 1$, the set*

$$\bigcup_{[L:k] \leq d} E(L)_{\text{tors}} \text{ is finite.}$$

Proof. Choose a prime ideal $\pi \subset k$ such that E has a good reduction at π . Let L be an extension of degree $\leq d$ over k . Extend π to L and denote its residue field by \tilde{L} . Then, since $[L:k] \leq d$, \tilde{L} is contained in the unique extension $\tilde{k}_{d!}$ of degree $d!$ over the residue field \tilde{k} of k at π . Note that for the global function field, a residue field has characteristic p which is the characteristic of k . So we consider two cases: n -torsion points where $p \nmid n$ and p^i -torsion points.

First, for any integer n not divisible by p , the n -torsion subgroup $E(L)[n]$ injects into $\tilde{E}(\tilde{k}_{d!})[n] \subset \tilde{E}(\tilde{k}_{d!})$ which doesn't depend on L . So there exists a constant c_1 such that $|E(L)[n]| \leq |\tilde{E}(\tilde{k}_{d!})| = c_1$, for any extension L of degree $\leq d$ over k .

Secondly, for the prime p , by [9, Lemma 1.1], we can choose a prime $\pi \subset k$ such that E has a good reduction at π and it induces an isomorphism between $E(\overline{K})[p^i]$ and $\tilde{E}(\tilde{k})[p^i]$ for all $i \geq 1$, where \overline{K} is an algebraic closure of K . So again by the same argument in the above, we can show that there exists a constant c_2 such that $|E(L)[p^i]| \leq c_2$ for all i and for any L of degree $\leq d$ over k , where c_2 does not depend on L but depends on the degree d . Therefore, $|E(L)_{\text{tors}}| \leq c_1 c_2$, for any extension L with $[L:k] \leq d$. This completes the proof. \square

Theorem 3.2. *Let E/k be an elliptic curve over a global function field with the field of constants \mathbb{F}_q , where q is a power of an odd prime p . Suppose E has split multiplicative reduction at ∞ . Then, for every automorphism $\sigma \in G_k$, the rank of $E(\overline{k}^\sigma)$ is infinite.*

Proof. Let A be the ring of functions in k regular outside ∞ . Let $\mathfrak{n} \cdot \infty$ be the conductor of E/k , where \mathfrak{n} is an ideal in A .

Since the characteristic p of k is not 2, we can write a Weierstrass equation of E/k in the form $y^2 = x^3 + ax^2 + bx + c$. By a change of variables, we may assume that a, b and c are in A .

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be all distinct primes dividing the ideal \mathfrak{n} in A . Fix $M := p_1 \cdots p_m$, where p_i is a non-zero prime element in \mathfrak{p}_i , for each $i = 1, \dots, m$. Consider the polynomial

$$f(x) := (1 + Mx)^3 + aM^2(1 + Mx)^2 + bM^4(1 + Mx) + cM^6 \in A[x].$$

Then, since k is a Hilbertian field by [8, Ch.14, Coro. 14.10], A is a Hilbertian ring. So there exists $m_1 \in A$ such that $f(m_1)$ is not a square in the completion k_∞ of k at ∞ , so in particular, it is not a square in k . Let $K_1 = k(\sqrt{f(m_1)})$. Then, ∞ does not split in K_1 so K_1 is an imaginary quadratic extension of k .

Next, by the Hilbert irreducibility over K_1 and by [8, Ch.11, Coro. 11.7], we can choose $m_2 \in A$ such that $f(m_2)$ is non-square in both K_1 and k_∞ . Let $K_2 = k(\sqrt{f(m_2)})$. Then K_1 and K_2 are linearly disjoint imaginary quadratic extensions over k .

By repeating this procedure over the composite field $K_1 K_1 \cdots K_n$ of imaginary quadratic extensions obtained from the previous steps inductively, we obtain an infinite sequence $\{K_i = k(\sqrt{f(m_i)})\}_{i=1}^\infty$ of quadratic extensions of k such that for all i ,

- (1) ∞ does not split in K_i , and
- (2) the fields K_i are pairwise linearly disjoint over k , (i.e. $[K_1 K_2 \cdots K_r : k] = 2^r$, for any $r \geq 1$).

Note that for every prime \mathfrak{p}_j dividing \mathfrak{n} ,

$$f(m_i) \equiv 1 \pmod{\mathfrak{p}_j}.$$

So this implies that all primes dividing \mathfrak{n} split in K_i for all i (see [15, Proposition 10.5]).

Let $\sigma \in G_k$. Then either $\sigma|_{K_i} = id_{K_i}$ for all i , or $\sigma|_{K_i} \neq id_{K_i}$ for some i .

First, suppose that for all i , $\sigma|_{K_i} = id_{K_i}$. Then, for each i , consider the element $\frac{1 + Mm_i}{M^2} \in k$. By plugging this into the given Weierstrass equation of E/k , we get

$$y^2 = \left(\frac{1 + Mm_i}{M^2}\right)^3 + a \left(\frac{1 + Mm_i}{M^2}\right)^2 + b \left(\frac{1 + Mm_i}{M^2}\right) + c = \frac{f(m_i)}{M^6}.$$

Hence, if we let

$$P_i = \left(\frac{1 + Mm_i}{M^2}, \frac{\sqrt{f(m_i)}}{M^3}\right),$$

then P_i is a point in $E(K_i)$ but it is not in $E(k)$. And moreover, since $K_i = K_i^\sigma$, P_i is fixed under σ .

So we get an infinite sequence $\{P_i\}_{i=1}^\infty$ of points in $E(\bar{k}^\sigma)$ such that each P_i is defined over the imaginary quadratic extension K_i over k . We may assume that these points P_i are not torsion points by Lemma 3.1. Now we show the points P_i are linearly independent. Suppose that they are dependent. Then, for some integers a_j ,

$$(***) \quad a_1 P_1 + a_2 P_2 + \cdots + a_r P_r = O.$$

Since the fields K_i are pairwise linearly disjoint over k , for each i , there is an automorphism of \bar{k} which fixes all but one K_i of K_1, \dots, K_r . Note that such an automorphism takes P_i to its inverse, $-P_i$. Applying this automorphism to $(***)$, we get

$$a_1 P_1 + \cdots + a_{i-1} P_{i-1} - a_i P_i + \cdots + a_r P_r = O.$$

By subtracting this from $(***)$, we get $2a_i P_i = O$, which implies $a_i = 0$ since the characteristic p of k is not 2 and P_i is not a torsion point. We conclude that the $P_i \in E(\bar{k})$ are linearly independent. Moreover, P_i are defined over the composite field of all quadratic field extensions of k , which is an abelian extension of k . Hence, the rank

of E over the maximal abelian extension of k in \bar{k}^σ is infinite, so the rank of $E(\bar{k}^\sigma)$ is infinite.

Next, suppose that there is an integer i such that $\sigma|_{K_i} \neq id_{K_i}$. Then, fix such a quadratic imaginary extension K_i , and call it K . Then, our construction shows that K satisfies the hypothesis of Proposition 2.3 (that is, (E, K) satisfies the Heegner hypothesis). So we complete the proof of this case as a consequence of Proposition 2.3. \square

4. MODULAR ELLIPTIC CURVES OVER TOTALLY REAL NUMBER FIELDS AND INFINITE RANK

In this section we treat the case of elliptic curves parametrized by certain Shimura curves.

Let F be a totally real number field with ring of integers \mathcal{O}_F . For any abelian group M we denote by $\hat{M} = M \otimes \prod_p \mathbb{Z}_p$ its profinite completion. The ring of adèles of F is denoted by \mathbb{A}_F .

Let $N \subset \mathcal{O}_F$ be a non-zero ideal, and let f be a newform on $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weight 2, level

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathcal{O}}_F) \mid c \in \hat{N} \right\},$$

trivial central character, and rational Hecke eigenvalues.

Then, (see [19]) there exists an elliptic curve E/F of conductor N such that

- (1) the L -functions of E and f coincide up to factors at primes dividing N , and
- (2) there exists a Shimura curve X/F and a surjective F -morphism

$$\pi : X \longrightarrow E.$$

We will refer to such elliptic curves as *modular* elliptic curves, for example all elliptic curves over $F = \mathbb{Q}$ are modular, by the celebrated results of Wiles et al [18, 17, 4].

Our second main result is the following.

Theorem 4.1. *Let E/F be a modular elliptic curve of conductor N over a totally real number field F . If either $[F : \mathbb{Q}]$ is odd, or N is non-trivial, then for each $\sigma \in G_F$, the rank of $E(\bar{F}^\sigma)$ is infinite.*

Our approach is similar to the function field case, and we first show that E has infinite rank over a suitable tower of ring class fields.

Let K/F be a totally imaginary quadratic extension, and denote by

$$\varepsilon = \otimes_\nu \varepsilon_\nu : F^\times \backslash \hat{F}^\times \longrightarrow \{\pm 1\}$$

the character associated to K/F . We say (E, K) satisfies the **weak Heegner Hypothesis** if

- (1) the relative discriminant of K/F is prime to N , and
- (2) $\varepsilon(N) = (-1)^{[F:\mathbb{Q}]-1}$.

Throughout this section, we let E/F be a modular elliptic curve with conductor N and K/F a totally imaginary quadratic extension such that (E, K) satisfies the weak Heegner hypothesis.

Proposition 4.2. *Let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime ideal not dividing $2N$ and satisfying $\varepsilon_{\mathfrak{p}}(N) = 1$, and denote by $K[\mathfrak{p}^n]$ the ring class field of K with conductor \mathfrak{p}^n . Then the rank of $E(K[\mathfrak{p}^n])$ is unbounded as $n \rightarrow \infty$.*

Let $K[\mathfrak{p}^\infty] = \bigcup_{n=1}^\infty K[\mathfrak{p}^n]$, and we start with the following lemma.

Lemma 4.3. *$E(K[\mathfrak{p}^\infty])$ has finite torsion.*

Proof. Let \mathfrak{q}_1 and \mathfrak{q}_2 be two principal primes of F which are inert in K/F and at which E has good reduction. Since they split completely in each $K[\mathfrak{p}^n]$, $K[\mathfrak{p}^\infty]$ has finite residue fields k_1 and k_2 at \mathfrak{q}_1 and \mathfrak{q}_2 , respectively. From good reduction follows that $E(K[\mathfrak{p}^\infty])_{\text{tors}}$ injects into $E(k_1) \oplus E(k_2)$, which is finite. \square

To prove Proposition 4.2 we need to construct suitable Heegner points on the Shimura curve X , for which we will need to introduce some notation. Our standard reference is the article of Zhang [19].

Fix a real place τ of F . Then there exists a unique quaternion algebra B which is ramified precisely at τ and at all the finite places ν with $\varepsilon_\nu(N) = -1$ (the cardinality of this set of places is appropriate, because we are assuming the weak Heegner hypothesis). We fix an embedding

$$\rho : K \hookrightarrow B.$$

Let $R \subset B$ be an order of type (N, K) , in other words R contains $\rho(\mathcal{O}_K)$ and has conductor N . The Shimura curve X/F corresponds to the Riemann surface

$$X(\mathbb{C}) \cong B_+ \backslash \mathbb{H} \times \hat{B}^\times / \hat{F}^\times \hat{R}^\times \cup \{\text{cusps}\},$$

where B_+ denotes the elements of B of totally positive reduced norm, \mathbb{H} denotes the complex upper half-plane, and $\{\text{cusps}\}$ is a finite set, which is non-empty only in the case where $F = \mathbb{Q}$ and $X = X_0(N)$.

For the construction of Heegner points it is more convenient to work with the Shimura curve Y corresponding to the Riemann surface

$$Y(\mathbb{C}) \cong B^\times \backslash \mathbb{H}^\pm \times \hat{B}^\times / \hat{R}^\times \cup \{\text{cusps}\},$$

of which X is a quotient by the action of \hat{F}^\times .

A point $z \in Y(\mathbb{C})$ is called a *CM point* if it is represented by an element of $\mathbb{H}^\pm \times \hat{B}$ of the form $(\sqrt{-1}, g)$. To a CM point z we associate the morphism

$$\phi_z = g^{-1} \rho g : K \longrightarrow \hat{B}.$$

The order $\text{End}(z) := \phi_z^{-1}(\hat{R})$ in K is called the *endomorphism ring* of z , and does not depend on the choice of g . It is of the form

$$\text{End}(z) = \mathcal{O}_F + c \mathcal{O}_K,$$

for an ideal $c \subset \mathcal{O}_F$ called the *conductor* of z .

Proof of Proposition 4.2. Let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime ideal as in the statement of the proposition. Denote by $F_{\mathfrak{p}}$ the completion of F at \mathfrak{p} with uniformizer ϖ . B splits at \mathfrak{p} , and we choose an isomorphism

$$B \otimes F_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$$

such that $\rho(\sqrt{-d}) \otimes 1$ in $\rho(K) \otimes F_{\mathfrak{p}}$ corresponds to the matrix $\begin{pmatrix} 0 & -1 \\ d & 0 \end{pmatrix} \in M_2(F_{\mathfrak{p}})$, where $K = F(\sqrt{-d})$, $d \in \mathcal{O}_F$.

Now let $P \in \hat{B}^\times$ be the element with \mathfrak{p} -component $\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$ and all other components 1. Let z_n be the CM point in $Y(\mathbb{C})$ corresponding to

$$(\sqrt{-1}, P^n) \in \mathbb{H}^\pm \times \hat{B}^\times.$$

As $\mathfrak{p} \nmid 2N$ we see that z_n has conductor \mathfrak{p}^n , i.e.

$$\text{End}(z_n) = \mathcal{O}_n := \mathcal{O}_F + \mathfrak{p}^n \mathcal{O}_K.$$

Denote by $x_n \in X(\mathbb{C})$ and $y_n \in E(\mathbb{C})$ the respective images of $z_n \in Y(\mathbb{C})$ under the maps

$$Y \longrightarrow X \xrightarrow{\pi} E.$$

We call the points y_n *Heegner points* (in contrast, Zhang only uses the term Heegner points for CM points with trivial conductor). Moreover, the points x_n , and thus also y_n , are defined over $K[\mathfrak{p}^n]$. In fact, by [19, §2.1.1] the set X_n of (positively oriented) CM points on X with conductor \mathfrak{p}^n is in bijection with $K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_n^\times \cong \text{Pic}(\mathcal{O}_n)$, with the action by $\text{Gal}(K[\mathfrak{p}^n]/K)$ given by class field theory.

By Lemma 4.3 and [12, Lemma 2.5], it suffices to show that the set

$$\{y_n \mid n = 1, 2, \dots\} \subset E(K[\mathfrak{p}^\infty])$$

is not finitely generated. Suppose it is finitely generated, then by the Mordell-Weil Theorem it is contained in $E(L)$ for some number field L , which we may suppose contains K . Let d be the degree of the F -morphism $\pi : X \rightarrow E$. Then $G_L = \text{Gal}(\bar{L}/L)$ acts on the fibers $\pi^{-1}(y_n)$, giving an upper bound for the G_L -orbit of x_n : $\#G_L \cdot x_n \leq d$. On the other hand, we have the lower bound $G_L \cdot x_n \geq \#\text{Pic}(\mathcal{O}_n)/[L : K]$, which is unbounded as $n \rightarrow \infty$. This contradiction completes the proof of Proposition 4.2. \square

Finally, the following proposition is analogous to Proposition 2.3, hence completes our second main result, Theorem 4.1, whose proof is given below.

Proposition 4.4. *Let $\sigma \in G_F$. Let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime ideal not dividing $2N$ and satisfying $\varepsilon_{\mathfrak{p}}(N) = 1$. Then, the rank of the Mordell-Weil group $E((K[\mathfrak{p}^n])^\sigma)$ over the fixed subfield of $K[\mathfrak{p}^n]$ under σ is unbounded, as n goes to ∞ . In particular, the rank of $E(K_{ab}^\sigma)$ is infinite, where K_{ab} is the maximal abelian extension of K .*

Proof. By generalizing the result in [12, Lemma 2.3] which is elementary class field theory, we note that the Galois group of a ring class field over K of conductor \mathfrak{p}^n over K is a product of p -cyclic groups, where p is the rational prime below \mathfrak{p} , and it has a dihedral group structure. So the proof is identical with the argument in Proposition 2.3 together with the unboundedness of the rank of $E(K[\mathfrak{p}^n])$ as n goes to infinity shown in Proposition 4.2. \square

We are now ready to prove Theorem 4.1. We need the following two lemmas.

Lemma 4.5. *Let K be a number field and τ_1, \dots, τ_m be a family of real embeddings of K . For $i = 1, 2, \dots, k$, let $f_i(x, y) \in K[x, y]$ be irreducible polynomials over $K(x)$. Let*

$H_K(f_i) = \{\alpha \in K : f_i(\alpha, y) \in K[y] \text{ is irreducible over } K\}$ be the Hilbert set of f_i over K . Then for any open interval I in \mathbb{R} ,

$$\left(\bigcap_{i=1}^k H_K(f_i) \right) \cap \left(\bigcap_{j=1}^m \tau_j^{-1}(I) \right) \neq \emptyset.$$

Proof. See [12, Lemma 1.2]. □

Lemma 4.6. *Let F be a totally real number field with real embeddings τ_j for $j = 1, \dots, n$. Then for every element $a \in F$ such that $\tau_j(a) < 0$ for all j , the field $F(\sqrt{a})$ is a totally imaginary quadratic extension over F .*

Proof. Let $a \in F$ such that $\tau_j(a) < 0$ for all j . First, if $\sqrt{a} \in F$, then $\tau_j(\sqrt{a}) \in \mathbb{R}$ for all j so its square $\tau_j(a)$ must be positive but this is not true by the assumption. So $\sqrt{a} \notin F$ and the field $F(\sqrt{a})$ is quadratic over F . Suppose $F(\sqrt{a})$ has a real embedding ρ . Then, the restriction of ρ to F is one of the real embeddings of F , so let $\rho|_F = \tau_k$ for some k . Also, since $\rho(\sqrt{a}) \in \mathbb{R}$,

$$0 < (\rho(\sqrt{a}))^2 = \rho(a) = \tau_k(a) < 0,$$

which leads a contradiction. So $F(\sqrt{a})$ has no real embedding, so it is totally imaginary. □

Proof of the main Theorem 4.1. Since F has characteristic 0, we fix a Weierstrass equation of E/F , $y^2 = x^3 + ax + b$. By a change of variables, we may assume that a and b are algebraic integers in F . Since F is a totally real number field, there exist finitely many real embeddings τ_j of F , for $j = 1, \dots, [F : \mathbb{Q}]$. Also, we let L be a quadratic field extension of F whose relative discriminant is N .

Let $M = p_1 p_2 \cdots p_k$, where p_i are prime elements in distinct prime ideals \mathfrak{p}_i in \mathcal{O}_F dividing N . Consider the polynomial

$$f(x) = (1 + Mx)^3 + aM^4(1 + Mx) + bM^6 \in \mathcal{O}_F[x].$$

For a polynomial g over F , denote by g^{τ_j} the polynomial obtained by applying τ_j to each coefficient of g .

Then, there exists a real number r such that for all $x < r$, the expressions $f^{\tau_j}(x)$ for all j are strictly negative. Let $I = (-\infty, r)$ be the open interval in \mathbb{R} of all real numbers less than r . Choose an integer $m_1 \in I \cap H_L(y^2 - f(x))$. Then, since $f^{\tau_j}(m_1) < 0$ for all j and $\sqrt{f(x)}$ is not in L , $K_{m_1} := \mathbb{Q}(\sqrt{f(m_1)})$ is an imaginary quadratic extension of F by Lemma 4.6 whose relative discriminant is different from that of L .

By Lemma 4.5, there exists a rational number in $I \cap (\bigcap_{j=1}^n H_{LK_{m_1}}(y^2 - f(x)))$. In particular, since the Hilbert set $H_{LK_{m_1}}(y^2 - f(x))$ contains infinitely many rational primes by [13, Chapter 9, Corollary 2.4], we can choose an integer $m_2 \in I \cap H_{LK_{m_1}}(y^2 - f(x))$. Then, $K_{m_2} := \mathbb{Q}(\sqrt{f(m_2)})$ is a quadratic imaginary extension of F by Lemma 4.6, and L , K_{m_1} and K_{m_2} are distinct, hence they are linearly disjoint over F . By repeating this procedure over the composite field $LK_{m_1}K_{m_2} \cdots K_{m_r}$ of imaginary quadratic extensions

obtained from the previous steps inductively, we obtain an infinite set S of integers such that for all $m \in S$,

- (1) $f^{\tau_j}(m) < 0$ for all j , so that $K_m := \mathbb{Q}(\sqrt{f(m)})$ is a totally imaginary quadratic extension of F ,
- (2) the fields in the infinite sequence $\{K_m\}_{m \in S}$ are linearly disjoint over F ,
(i.e. $[K_{m_1}K_{m_2} \cdots K_{m_r} : \mathbb{Q}] = 2^r$, for any $m_i \in S$ and for every $r \geq 1$)
- (3) the relative discriminant of K_m/F is different from N and
- (4) $f(m) \equiv 1 \pmod{\mathfrak{p}_i}$ for distinct prime ideals \mathfrak{p}_i in \mathcal{O}_F dividing N .

Then for each $m \in S$, (E, K_m) satisfies the first weak Heegner hypothesis by (3) as above and it satisfies the second weak Heegner hypothesis by (4) and by the assumption on $[F : \mathbb{Q}]$ or N .

So for $\sigma \in G_F$, depending on whether σ fixes K_m for all $m \in S$ or not, the rest of the proof is the same as [12, Theorem 1.3] (or our first main result, Theorem 3.2) by using Proposition 4.4. \square

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